

FLUX QUANTIZATION FOR A SUPERCONDUCTING RING IN THE SHAPE OF A MÖBIUS BAND

JONATHAN ROSENBERG AND YEHOSHUA DAN AGASSI

ABSTRACT. We give two derivations of magnetic flux quantization in a superconducting ring in the shape of a Möbius band, one using direct study of the Schrödinger equation and the other using the holonomy of flat $U(1)$ -gauge bundles. Both methods show that the magnetic flux must be quantized in integral or half-integral multiples of $\Phi_0 = hc/(2e)$. Half-integral quantization shows up in “nodal states” whose wavefunction vanishes along the center of the ring, for which there is now some experimental evidence.

One of the best-known macroscopically observable quantum effects is the quantization of magnetic flux through a superconducting ring M in units of $\Phi_0 = \frac{hc}{2e}$ [2]. The question we treat here is this: What happens to this condition if the superconducting ring M is in the shape of a Möbius band (Figure 1) rather than an annulus? Does the condition remain the same, or should it be modified? This provides a good test case for the application of topology in physics. It should be possible to check the result experimentally since crystals of potentially superconducting materials such as NbSe_3 have recently been produced with a Möbius band shape [9, 10]. Investigation of a different topology, the “figure eight,” was carried out by Vodolazov and Peeters [11].

Several possible derivations for flux quantization have been given. We shall analyze what happens to them under this “exotic” topology, under the assumption that the superconductor is extremely thin and can be treated as if it were 2-dimensional. We can take coordinates in the superconductor to be x and y , with x (going around the ring M) going from 0 to ℓ , y going from $-w$ to w , and the point $(0, y)$ identified with $(\ell, -y)$. The simplest treatment (following [7], based roughly on the classic paper [1]) starts with the Schrödinger equation for a Cooper pair of charge $-2e$ in an electromagnetic field with circular symmetry. Recall that the vector potential \mathbf{A} is not necessarily globally defined on the ring, though we can take it to be single-valued on $[0, \ell] \times [-w, w]$, possibly with different values at $(0, y)$ and at $(\ell, -y)$. By the Meissner effect, $\mathbf{B} = \nabla \times \mathbf{A} = \mathbf{0}$ in the superconductor

2010 *Mathematics Subject Classification.* Primary 82D55; Secondary 53C05.

Key words and phrases. Möbius band, superconductor, flux quantization, $U(1)$ -gauge bundle.

JR partially supported by NSF Grants DMS-0805003 and DMS-1206159.

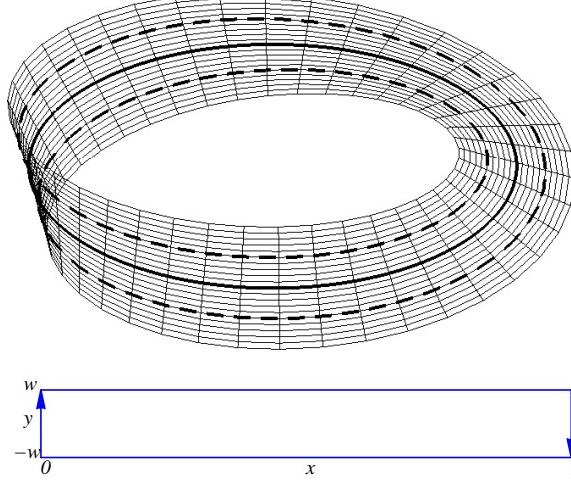


FIGURE 1. A Möbius band with its coordinate system

M , which implies that \mathbf{A} is *locally* of the form $\nabla\phi$. Again we can take ϕ to be single-valued on $[0, \ell] \times [-w, w]$, possibly with different values at $(0, y)$ and at $(\ell, -y)$. The (time-independent) Schrödinger equation becomes

$$(0.1) \quad \frac{1}{2m} \left(-i\hbar\nabla + \frac{2e}{c} \mathbf{A} \right)^2 \psi + V\psi = E\psi,$$

where ψ is the wavefunction of a Cooper pair, treated a single boson with mass m . Let ψ_0 be a solution of (0.1) with $\mathbf{A} = \mathbf{0}$. Then if $\psi = \psi_0 e^{-i\alpha\phi}$ for a constant α , we have

$$\begin{aligned} \left(-i\hbar\nabla + \frac{2e}{c} \mathbf{A} \right) \psi &= \left(-i\hbar\nabla\psi_0 \right) e^{-i\alpha\phi} - \hbar\alpha\psi_0 e^{-i\alpha\phi} \mathbf{A} + \frac{2e}{c} \psi_0 e^{-i\alpha\phi} \mathbf{A} \\ &= -i\hbar e^{-i\alpha\phi} \nabla\psi_0, \end{aligned}$$

provided that $\alpha = \frac{2e}{\hbar c}$. Thus ψ will be a solution of (0.1) for this value of α . Since the wavefunction must be well-defined globally, in a ground state where ψ_0 is everywhere non-zero on the interior of M , $\phi(\ell, -y) - \phi(0, y)$ must have a constant value of $\frac{2\pi n\hbar c}{2e} = n\frac{\hbar c}{2e} = n\Phi_0$, for some $n \in \mathbb{Z}$, independent of y . In particular, if C is the closed curve through the middle of the superconductor, corresponding to $y = 0$ in our coordinate system (see solid black curve in Figure 1), then $\oint_C \mathbf{A} \cdot d\mathbf{r} = n\Phi_0$, which is the flux quantization condition. Note however that if we take the curve C' given by $y = \text{const.}$ with the constant non-zero, then when x runs from 0 to ℓ , we end up on the opposite side of C from where we started, and so one has to let x run all the way out to 2ℓ to get back to the starting point (see dashed black curve in Figure 1). Thus for C' , we have the modified quantization condition $\oint_{C'} \mathbf{A} \cdot d\mathbf{r} = 2n\Phi_0$, with an extra factor of 2.

However, another phenomenon, proposed on different theoretical grounds (calculations with the Ginzburg-Landau and Bogoliubov-de Gennes models) in [4, 5], and supported by experimental evidence in [6], is also possible. Namely, one can have a “nodal” state supported near the dashed curve C' , with $\psi_0 = 0$ on the circle C given by $y = 0$. Then it suffices to for $\phi(\ell, y) - \phi(0, y)$ to be a *half-integer* multiple of Φ_0 . This will still give a globally single-valued wavefunction since the point (ℓ, y) is identified with $\phi(0, -y)$ and $\phi(\ell, -y) - \phi(0, -y)$ is again a half-integer. Thus the flux appears to be quantized in half-integral units of Φ_0 , as was observed in [6]. A slightly different analysis by Mila *et al.* [8] showed that for a Möbius “ladder” geometry, quantization should be in multiples of Φ_0 “as long as coherent motion between the chains is possible,” and in multiples of $\Phi_0/2$ when there is no coherent motion between the chains (so that particles effectively travel along the curve C').

A more sophisticated approach follows the ideas of [7]. Following Dirac [3] and Weyl, we view the magnetic field as a $U(1)$ -gauge field. More precisely, the vector potential \mathbf{A} is a connection on a $U(1)$ -bundle and the field strength (curvature 2-form) is the magnetic field. The *phase* of the wave function ψ is a section of this $U(1)$ -bundle [3]. Because of the Meissner effect, the curvature of the bundle vanishes on M , i.e., the bundle is flat. Since $H^2(M, \mathbb{Z}) = 0$, the bundle is also topologically trivial, and the connection differs from the usual connection (corresponding to the case $\mathbf{A} = \mathbf{0}$) by a constant 1-form, which we can identify with a single real number, which is the value of the holonomy or flux $\Phi = \oint_C \mathbf{A} \cdot d\mathbf{r}$. Since the wavefunction must be single-valued on M , and since as we saw above, the change in the wavefunction as we go around the loop C is

$$e^{-i \frac{2e}{\hbar c} \Phi},$$

we obtain the flux quantization condition $\Phi = n\Phi_0$.

The “nodal state” case can be treated similarly, except that we replace M by the complement of C since we are assuming the Cooper pairs are localized away from the center of the Möbius band. (If the wave function vanishes on C , then its phase there is not well-defined, so the bundle is not defined on C .) Since the inclusion $(M \setminus C) \hookrightarrow M$ sends a generator of the fundamental group of $M \setminus C$ to *twice* a generator of the fundamental group of M , we have in effect twice as many possible flat bundles. In other words, we only require $\oint_{C'} \mathbf{A} \cdot d\mathbf{r}$ to be integral, which means that flux around the superconducting loop satisfies the flux quantization condition $\Phi = n\Phi_0$ with n a half-integer.

The second approach also explains the answer to another question: the flux $\Phi = \oint_C \mathbf{A} \cdot d\mathbf{r}$ only depends on the homology class of the loop C in $H_1(M, \mathbb{Z}) \cong \pi_1(M) \cong \mathbb{Z}$, since this is a basic fact about holonomy of flat connections. In particular, since C' is homologous to $2C$, the flux around C' is quantized in twice the units of the flux around C . From the first point of view, this is a bit harder to see, since if C_1 and C_2 are homologous loops,

the usual argument (when M is an annulus) would be to take the region D with boundary $C_1 - C_2$ (i.e., $C_1 \cup C_2$, but with reversed orientation on C_2) and to use Stokes' Theorem to argue that

$$(0.2) \quad 0 = \iint_D (\nabla \times \mathbf{A}) \cdot \mathbf{n} = \oint_{C_1} \mathbf{A} \cdot d\mathbf{r} - \oint_{C_2} \mathbf{A} \cdot d\mathbf{r}.$$

This runs into difficulties since M is not orientable (so that the normal \mathbf{n} is not well defined), and thus Stokes' Theorem doesn't seem to apply. But one can rectify things by cutting M open along a curve whose complement is orientable.

An issue not settled by our analysis is what happens with a “thick” Möbius band (thickness being measured compared to the penetration depth), where 2-dimensionality is no longer a good assumption. In this case, since there may not be nodal states in the sense described above, one might expect flux quantization only in integral multiples of Φ_0 . It would be interesting to test for this experimentally, as the results of [6] are only for a thin film.

REFERENCES

- [1] N. Byers and C. N. Yang, Theoretical considerations concerning quantized magnetic flux in superconducting cylinders, *Phys. Rev. Lett.* **7** (1961), 46–49.
- [2] Bascom S. Deaver, Jr., and William M. Fairbank, Experimental evidence for quantized flux in superconducting cylinders, *Phys. Rev. Lett.* **7** (1961), 43–46.
- [3] P. A. M. Dirac, Quantised singularities in the electromagnetic field, *Proc. Royal Soc. London A* **133** (1931), no. 821, 60–72.
- [4] M. Hayashi and H. Ebisawa, Little-Parks oscillation of superconducting Möbius strip, *J. Phys. Soc. Japan* **70** (2001), 3495–3498.
- [5] M. Hayashi, H. Ebisawa, and K. Kuboki, Superconductivity on a Möbius strip: Numerical studies of order parameter and quasiparticles, *Phys. Rev. B* **72** (2005), 024505 (7 pp.), arXiv:cond-mat/0502149.
- [6] M. Hayashi, H. Ebisawa, and K. Kuboki, Superconductivity and charge-density wave in ring- or Moebius-shaped NbSe₃ and TaS₃ single crystals, *Acta Cryst. A* **64** (2008), C508–C509.
- [7] B. T. McInnes, Remarks on the geometric interpretation of superconductive flux quantisation, *J. Physics A: Math. Gen.* **17** (1984), 3101–3105.
- [8] F. Mila, C. A. Stafford, and S. Capponi, Persistent currents in a Möbius ladder: A test of interchain coherence of interacting electrons, *Phys. Rev. B* **57** (1998), 1457–1460.
- [9] S. Tanda, T. Tsuneta, Y. Okajima, K. Inagaki, K. Yamaya, and N. Hatakenaka, A Möbius strip of single crystals, *Nature* **417** (2002), 397–398.
- [10] S. Tanda, T. Tsuneta, T. Toshima, T. Matsuura, and M. Tsubota, Topological crystals, *J. de Physique IV* **131** (2005), 289–294.
- [11] D. Y. Vodolazov and F. M. Peeters, Stable and metastable states in a mesoscopic superconducting “eight” loop in presence of an external magnetic field, *Physica C* **400** (2004), 165–170.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD
20742, USA

E-mail address: jmr@math.umd.edu

URL: <http://www.math.umd.edu/~jmr>

NAVAL SURFACE WARFARE CENTER, CARDEROCK DIVISION, 9500 MACARTHUR BLVD.,
WEST BETHESDA, MD 20817, USA

E-mail address: yehoshua.agassi@navy.mil